Essential Trigonometric Formulas

A concise school reference guide

The unit circle, identities, transformations and trigonometric equations

1. Degrees and radians

A full rotation:

$$360^\circ = 2\pi \text{ rad}.$$

A half-turn:

$$180^\circ = \pi \text{ rad}.$$

Converting degrees to radians:

$$\alpha_{\text{rad}} = \alpha_{\text{deg}} \cdot \frac{\pi}{180}.$$

Converting radians to degrees:

$$\alpha_{\text{deg}} = \alpha_{\text{rad}} \cdot \frac{180}{\pi}.$$

2. Definitions of the trigonometric functions

On the unit circle, the point corresponding to an angle α has coordinates

$$(\cos\alpha, \sin\alpha).$$

Tangent:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cos\alpha \neq 0.$$

Cotangent:

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}, \qquad \sin\alpha \neq 0.$$

3. Key exact values

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
α	0°	30°	45°	60°	90°
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined
$\cot\alpha$	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

4. Fundamental trigonometric identities

The Pythagorean identity:

$$\sin^2 \alpha + \cos^2 \alpha = 1.$$

Relationship between tangent and cosine:

$$1 + \tan^2 \alpha = \frac{1}{\cos^2 \alpha}, \qquad \cos \alpha \neq 0.$$

Relationship between cotangent and sine:

$$1 + \cot^2 \alpha = \frac{1}{\sin^2 \alpha}, \qquad \sin \alpha \neq 0.$$

Relationship between tangent and cotangent:

$$\tan \alpha \cdot \cot \alpha = 1,$$

provided that both functions are defined.

5. Parity and periodicity

Sine, tangent and cotangent are odd functions:

$$\sin(-\alpha) = -\sin \alpha,$$

$$\tan(-\alpha) = -\tan \alpha,$$

$$\cot(-\alpha) = -\cot \alpha.$$

Cosine is an even function:

$$\cos(-\alpha) = \cos \alpha.$$

The period of sine and cosine is 2π:

$$\sin(\alpha + 2\pi k) = \sin \alpha,$$

$$\cos(\alpha + 2\pi k) = \cos \alpha.$$

The period of tangent and cotangent is π:

$$\tan(\alpha + \pi k) = \tan \alpha,$$

$$\cot(\alpha + \pi k) = \cot \alpha, \qquad k \in \mathbb{Z}.$$

6. Signs of the functions by quadrant

Quadrant	I	II	III	IV
$\sin\alpha$	$+$	$+$	$-$	$-$
$\cos\alpha$	$+$	$-$	$-$	$+$
$\tan\alpha$	$+$	$-$	$+$	$-$
$\cot\alpha$	$+$	$-$	$+$	$-$

7. Angle addition and subtraction formulas

Sine of a sum:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Sine of a difference:

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Cosine of a sum:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Cosine of a difference:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Tangent of a sum:

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

Tangent of a difference:

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

These formulas are valid whenever all expressions involved are defined.

8. Double-angle formulas

Sine of a double angle:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

Cosine of a double angle:

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha.$$

Equivalent forms:

$$\cos(2\alpha) = 2\cos^2\alpha - 1,$$

$$\cos(2\alpha) = 1 - 2\sin^2\alpha.$$

Tangent of a double angle:

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha},$$

provided that both sides are defined.

9. Half-angle formulas

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}.$$

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}.$$

When taking the square root, choose the sign according to the quadrant containing $\frac{\alpha}{2}$:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}.$$

Tangent of a half-angle:

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}.$$

Also:

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}.$$

These formulas are used only when their denominators are non-zero.

10. Sum-to-product formulas

Sum of sines:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

Difference of sines:

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

Sum of cosines:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

Difference of cosines:

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

11. Product-to-sum formulas

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta).$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta).$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta).$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta).$$

12. Related-angle identities

If an angle contains

$$\frac{\pi}{2} \pm \alpha \quad \text{or} \quad \frac{3\pi}{2} \pm \alpha,$$

the function changes to its cofunction:

$$\sin \longleftrightarrow \cos, \qquad \tan \longleftrightarrow \cot\,.$$

If an angle contains

$$\pi \pm \alpha \quad \text{or} \quad 2\pi \pm \alpha,$$

the name of the function remains unchanged.
The sign of the result is determined by the quadrant in which the original angle lies.
Examples:

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha.$$

$$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha.$$

$$\sin(\pi - \alpha) = \sin\alpha.$$

$$\cos(\pi - \alpha) = -\cos\alpha.$$

$$\tan(\pi + \alpha) = \tan\alpha.$$

$$\cot(\pi - \alpha) = -\cot\alpha.$$

13. Inverse trigonometric functions

Arcsine:

$$y = \arcsin x \quad \Longleftrightarrow \quad \sin y = x,$$

$$-1 \le x \le 1, \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

Arccosine:

$$y = \arccos x \quad \Longleftrightarrow \quad \cos y = x,$$

$$-1 \le x \le 1, \qquad 0 \le y \le \pi.$$

Arctangent:

$$y = \arctan x \quad \Longleftrightarrow \quad \tan y = x,$$

$$-\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Arccotangent, using the convention $0 < y < \pi$:

$$y = \operatorname{arccot} x \quad \Longleftrightarrow \quad \cot y = x,$$

$$0 < y < \pi.$$

14. Basic trigonometric equations

The equation

$$\sin x = a$$

has real solutions only when

$$|a| \leq 1.$$

Its general solution is

$$x = (-1)^n \arcsin a + \pi n, \qquad n \in \mathbb{Z}.$$

The equation

$$\cos x = a$$

has real solutions only when

$$|a| \leq 1.$$

Its general solution is

$$x = \pm \arccos a + 2\pi n, \qquad n \in \mathbb{Z}.$$

The equation

$$\tan x = a$$

has the general solution

$$x = \arctan a + \pi n, \qquad n \in \mathbb{Z}.$$

The equation

$$\cot x = a$$

has the general solution

$$x = \operatorname{arccot} a + \pi n, \qquad n \in \mathbb{Z}.$$

Special cases:

$$\sin x = 0 \quad \Longrightarrow \quad x = \pi n.$$

$$\cos x = 0 \quad \Longrightarrow \quad x = \frac{\pi}{2} + \pi n.$$

$$\tan x = 0 \quad \Longrightarrow \quad x = \pi n.$$

$$\cot x = 0 \quad \Longrightarrow \quad x = \frac{\pi}{2} + \pi n.$$

15. Tangent half-angle substitution

Introduce a new variable:

$$t = \tan\frac{x}{2}.$$

Then:

$$\sin x = \frac{2t}{1+t^2}.$$

$$\cos x = \frac{1-t^2}{1+t^2}.$$

$$\tan x = \frac{2t}{1-t^2}.$$

Also:

$$\cot x = \frac{1-t^2}{2t}.$$

This substitution converts rational expressions in

$$\sin x \quad \text{and} \quad \cos x$$

into rational expressions in the variable t.
When using the substitution, the values

$$x = \pi + 2\pi n, \qquad n \in \mathbb{Z},$$

must be checked separately, because $\tan\frac{x}{2}$ is undefined at these points.

Before giving your answer

Check:

- whether the angle is measured in degrees or radians;
- the domain of every expression;
- the signs of the functions in each quadrant;
- the period of the relevant function;
- whether all solutions lie in the required interval.

This reference sheet is intended for quick review. Before using a formula, check its domain and the units in which the angles are measured.